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On Proximinality and Sets of Operators. II. Nonexistence of Best Approximation from the Sets of Finite Rank Operators

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The most important result in this paper is that the set $K_n(l_1, c_0)$ is not proximinal in $L(l_1, c_0)$. This gives a negative solution to Problem 5.2.1 and a positive solution to Problem 5.2.4 of Deutsch, Mach, and Saatkamp (*J. Approx. Theory* **33** (1981), 199–213).

INTRODUCTION

If A is a closed subset of the normed linear space X, then A is said to be "proximinal" in X if for each $x \in X$, there is $y_0 \in A$ such that

$$\|x - y_0\| = d(x, A) = \inf\{\|x - y\|; y \in A\}.$$

In this case y_0 is called a "best approximation" to x from A. If B is a subset of X then

$$\delta(B, A) = \sup\{d(x, A); x \in B\},$$

is the deviation of B from A, and

$$d_n(B, X) = \inf\{\delta(B, N); N \text{ is an } n\text{-dimensional subspace of } X\},$$

is the Kolmogrov n-width of B in X.

If X and Y are two normed linear spaces, then $L(X, Y)$ denotes the set of all bounded linear operators from X to Y, $K(X, Y)$ the set of all compact operators in $L(X, Y)$, and $K_n(X, Y)$ the set of all operators in $L(X, Y)$ of rank $\leqslant n$.

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The proximinality of $K_n(X, Y)$ in $L(X, Y)$ and $K(X, Y)$ has been studied by several authors. It is known that if Y^* is the dual space of Y then $K_n(X, Y^*)$ is proximinal in $L(X, Y^*)$. The more interesting problem turns out to be the proximinality of $K_n(X, Y)$ in $K(X, Y)$ and $L(X, Y)$ when $Y = C_0(Q)$, for arbitrary locally compact Hausdorff space Q. Deutsch *et al.* [2] proved that, if X^* is strictly convex then $K_n(X, C_0(Q))$ is proximinal in $K(X, C_0(Q))$, and Kamal [4] proved that, if X^* is uniformly convex then $K_n(X, C_0(Q))$ is proximinal in $L(X, C_0(Q))$.

This paper contains a further study for the proximinality of $K_n(X, C_0(Q))$ in $K(X, C_0(Q))$ and $L(X, C_0(Q))$. In Section 1, it is shown that for each positive integer $n \geqslant 1$, the set $K_n(l_1, c_0)$ is not proximinal in $L(l_1, c_0)$. This gives a negative solution to Problem 5.2.1 of Deutsch *et al.* [2]. Since by Mach and Ward [5], $K(l_1, c_0)$ is proximinal in $L(l_1, c_0)$, it follows that the solution of Problem 5.2.4 of Deutch *et al.* [2] is positive, that is there are Banach spaces X and Y such that $K_n(X, Y)$ is not proximinal in $L(X, Y)$, whereas $K(X, Y)$ is proximinal in $L(X, Y)$.

The Hausdorff space Q will be said to *contain* Q_0 if it contains an infinite convergent sequence of distinct elements. In Section 2, it is shown that if Q contains Q_0. then $K_n(l_1, C(Q))$ is not proximinal in $K(l_1, C(Q))$. This shows that generally it is not necessary that $K_n(X, Y)$ is proximinal even in $K(X, Y)$. Brown [1] proved that if Q satisfies a certain condition then there is a bounded subset A in an $(n+3)$-dimensional subspace of $C(Q)$, such that the n-width of A, $d_n(A, C(Q))$ is not attained. Although c_0 is not isometric to any $C(Q)$ for which Q satisfies the given condition, it is shown that for each positive integer $n \geqslant 1$, there is a bounded subset A of c_0, such that the n-width of A, $d_n(A, c_0)$ is not attained. Since it is easy to show that for each relatively compact subset K of c_0, the n-width $d_n(K, c_0)$ is attained, it follows that this result cannot be improved in c_0.

The rest of the Introduction will cover some definitions and known results, that will be used frequently in this paper. If Q is a Hausdorff topological space, X is a normed linear space and τ is a topology defined on X, then $C(Q, (X, \tau))$ denotes the set of all bounded functions from Q to X, which are continuous with respect to τ. If $\tau = \|\cdot\|$, then $C_0(Q, X) = \{f \in C(Q, (X, \|\cdot\|));\ \forall \varepsilon > 0$ the set $\{q \in Q;\ \|f(q)\| \geqslant \varepsilon\}$ is compact$\}$. If $X = R$ the set of real numbers, then $C_0(Q, R)$ is denoted by $C_0(Q)$. If X^* is the dual space of X then

$$C_0(Q, (X^*, \omega^*)) = \{f \in C(Q, (X^*, \omega^*));\ \hat{x} \circ f \in C_0(Q)\ \forall x \in X\},$$

where $\hat{x}$ is the image of x under the canonical injection of X in X^{**}.

As a special case if Q is the set of all positive integers, then $C_0(Q, X)$ consists of all bounded sequences in X that converge to zero, and will be denoted by $c_0(X)$.

The importance of introducing the Banach space $C_0(Q, X)$ can be seen in

0.1. LEMMA. *Let X be a Banach space, Q a locally compact Hausdorff space, and for each nonnegative integer n, let*

$$C_n = \bigcup_N \{C_0(Q, N);\ N \text{ is an } n\text{-dimensional subspace of } X^*\}.$$

The set $K_n(X, C_0(Q))$ is proximinal in $L(X, C_0(Q))$ [resp. $K(X, C_0(Q))$] if and only if, for each $f \in C_0(Q, (X^*, \omega^*))$ [resp. $f \in C_0(Q, X^*)$], there is an n-dimensional subspace N_0 of X^*, and $g \in C_0(Q, N_0)$ such that

$$\|f - g\| = d(f, C_n).$$

The proof of this lemma can be obtained from Deutsch et al. [2], *and can be found in Kamal* [4].

0.2. DEFINITION. Let X be a Banach space, Q a locally compact Hausdorff space, and let C_n be as in Lemma 0.1.

(a) For each $f \in C_0(Q, (X^*, \omega^*))$ let $a_n(f)$ denotes $d(f, C_n)$.

(b) For each $T \in L(X, C_0(Q))$ let $a_n(T)$ denotes $d(T, K_n(X, C_0(Q)))$.

It is obvious from Lemma 0.1 that there is no problem in introducing the same symbol "a_n" in both cases of Definition 0.2, since $a_n(f)$ is attained for each $f \in C_0(Q, (X^*, \omega^*))$ [resp. $f \in C_0(Q, X^*)$], if and only if $a_n(T)$ is attained for each $T \in L(X, C_0(Q))$ [resp. $T \in K(X, C_0(Q))$].

1. $K_n(l_1, c_0)$ Is Not Proximinal in $L(l_1, c_0)$

In this section it will be shown that for each positive integer $n \geqslant 1$, the set $K_n(l_1, c_0)$ is not proximinal in $L(l_1, c_0)$. After referring to Lemma 0.1, "taking Q to be the set of positive integers," it is enough to construct for each $n \geqslant 1$, a bounded sequence in l_∞, such that $\eta_i \to^{\omega^*} 0$ and $a_n(\{\eta_i\}_{i=1}^\infty)$ is not attained. The main steps in the proof are to construct in $l_{2^n+1}^\infty$ a finite subset satisfying a certain condition (Lemma 1.2), the by injecting this set in a certain way in l_∞ (Lemma 1.3) a bounded sequence $\{\eta_i\}_{i=1}^\infty$ will be constructed in l_∞ with the required properties.

1.1. LEMMA. *Let n be a fixed positive integer, $m = 2^n$, Λ the set of all $\sigma = (\sigma_1, ..., \sigma_n) \in l_n^\infty$ with $|\sigma_i| = 1$ for $i = 1, 2, ..., n$ and let $A = (a_{ij})_{(i,j)=(1,1)}^{(n,m)}$ be the matrix in which the rows are the element of Λ. Let $b_1, ..., b_n$ be the columns of A, $z_i' = 2b_i$ for $i = 1, 2, ..., n$ and $\gamma = (1, 1, ..., 1) \in l_m^\infty$. Then*

(1) $d_{n-1}(\{z_1', ..., z_n'\}, l_m^\infty) = 2$.

(2) *For any number a, with $0 \leqslant a \leqslant 2$*

$$d_n(\{z_1', ..., z_n', a\gamma\}, l_m^\infty) = a.$$

Proof. (2) Let A be the balanced convex hull of $\{z'_1,\dots, z'_n, a\gamma\}$, and let $F_r(A)$ be the boundary of A. By Brown [1] $d_n(A, l_m^\infty) = \inf\{\|x\|; x \in F_r(A)\}$, thus $d_n(A, l_m^\infty) \leqslant a$. Second, let $x = \sum_{i=1}^n \alpha_i z'_i + \alpha_{n+1} a\gamma \in F_r(A)$, by the construction of the z'_i, s, and γ

$$\begin{aligned}\|x\| = \left\| \sum_{i=1}^n \alpha_i z'_i + \alpha_{n+1} a\gamma \right\| &= 2 \sum_{i=1}^n |\alpha_i| + a\,|\alpha_{n+1}| \\ &= 2 \sum_{i=1}^{n+1} |\alpha_i| + (a-2)\,|\alpha_{n+1}| \\ &= 2 + (a-2)\,|\alpha_{n+1}| \\ &= 2(1 - |\alpha_{n+1}|) + a\,|\alpha_{n+1}| \\ &\geqslant \min\{2, a\} = a. \quad \blacksquare\end{aligned}$$

In Lemma 1.2, for x and y in l_m^∞ and a, b in R, let $(ax, by) = (ax_1, ax_2,\dots, ax_m, by_1, by_2,\dots, by_m) \in l_m^\infty \times l_m^\infty = l_{2m}^\infty$. Conversely if $z = (z_1,\dots, z_{2m}) \in l_{2m}^\infty$ let;

$$P_1\colon\ l_{2m}^\infty \to l_m^\infty,$$
$$P_1(z) = (z_1,\dots, z_m)$$

and

$$P_2\colon\ l_{2m}^\infty \to l_m^\infty,$$
$$P_2(z) = (z_{m+1},\dots, z_{2m}).$$

Clearly if F is an n-dimensional subspace of l_{2m}^∞ then $P_1(F)$ and $P_2(F)$ are subspaces of l_m^∞, each of dimension less than or equal to n.

1.2. LEMMA. *Let $z'_1,\dots, z'_n, \gamma$, n and m be as in Lemma 1.1. Let $Z_i = (z'_i, z'_i) \in l_{2m}^\infty$, $\phi = (\theta_1\gamma, \theta_2\gamma)$ and $\Psi = (\psi_1\gamma, \psi_2\gamma)$, where $\{\psi_1, \psi_2, \theta_1, \theta_2\}$ satisfies the following conditions.*

(1) $\theta_1 = 2$ *and* $\psi_1 > 1$.

(2) $\theta_2\psi_2 < 0$, $|\psi_2| > |\theta_2|$ *and* $|\psi_2| + |\theta_2| \geqslant 2$.

Let F be an $(n+1)$ dimensional subspace of l_{2m}^∞ such that

$$\delta(\{Z_1, Z_2,\dots, Z_n, \Phi, \Psi\}, F) \leqslant 1.$$

If $\beta \in F$ and $\|\Psi - \beta\| \leqslant 1$ then $\|\beta\| \geqslant 1$.

Proof. Let $\{x_1,\dots, x_n, \alpha, \beta\} \subseteq F$ be so that $\|Z_i - X_i\| \leqslant 1$ for $i = 1, 2,\dots, n$, $\|\Phi - \alpha\| \leqslant 1$ and $\|\Psi - \beta\| \leqslant 1$, and let M be the subspace of F generated by

$\{x_1, \dots, x_n\}$. By Lemma 1.1 dim $M = n$, so there are two real numbers a_1 and a_2 such that $|a_1| + |a_2| = 1$ and $a_1\alpha + a_2\beta \in M$. By Lemma 1.1(2) for $i = 1, 2$,

$$\begin{aligned}|\theta_i a_1 + \psi_i a_2| &\leqslant d((\theta_i a_1 + \psi_i a_2)\gamma, P_i(M)) \\ &\leqslant \|(\theta_i a_1 + \psi_i a_2)\gamma - P_i(a_1\alpha + a_2\beta)\| \\ &\leqslant |a_1|\ \|P_i(\Phi) - P_i(\alpha)\| + |a_2|\ \|P_i(\Psi) - P_i(\beta)\| \\ &\leqslant |a_1| + |a_2| \\ &= 1.\end{aligned}$$

Since $\theta_1 = 2$ and $\psi_1 > 1$, the case $i = 1$ gives $a_1 a_2 < 0$, and since $\psi_2\theta_2 < 0$, the case $i = 2$ gives the inequality $|\theta_2|\ |a_1| + |\psi_2|\ \ |a_2| \leqslant 1$, so $(|\theta_2| - |\psi_2|)\ |a_1| + |\psi_2| \leqslant 1$, but $|\psi_2| > |\theta_2||$ so

$$|a_1| \geqslant \frac{|\psi_2| - 1}{|\psi_2| - |\theta_2|} \geqslant \frac{|\psi_2| - ((|\psi_2| + |\theta_2|)/2)}{|\psi_2| - |\theta_2|} = \frac{1}{2}.$$

Therefore, $|a_1| \geqslant |a_2|$. Also by Lemma 1.1

$$\begin{aligned}2 = \theta_1 \leqslant d(\theta_1\gamma\, P_1(M) &\leqslant \left\| \theta_1\gamma - P_1 \frac{(a_1\alpha + a_2\beta)}{a_1} \right\| \\ &\leqslant \|P_1(\Phi) - P_1(\alpha)\| + \left|\frac{a_2}{a_1}\right|\ \|P_1(\beta)\| \\ &\leqslant 1 + \left|\frac{a_2}{a_1}\right|\ \|\beta\|.\end{aligned}$$

So $\|\beta\| \geqslant (\theta_1 - 1)\ |a_1/a_2| \geqslant 1$. ∎

In Lemma 1.3 l_∞ will be considered as $\prod_{i=1}^{\infty} l_{2m}^{\infty}$, where $m = 2^n$, that is, $(y_1, y_2, \dots) \in \prod_{i=1}^{\infty} l_{2m}^{\infty}$ means that $y_i \in l_{2m}^{\infty}$.

1.3. Lemma. *Let $Z_1, \dots, Z_n$, γ, and m be as in Lemma* 1.2, *and for each positive integer $k \geqslant 1$ let*;

$$\Phi_k = \left(2\gamma, \frac{-(k-1)}{k}\gamma\right) \in l_{2m}^{\infty},$$

and

$$\Psi_k = \left(\frac{k+1}{k}\gamma, \frac{k+1}{k}\gamma\right) \in l_{2m}^{\infty}.$$

Define the sequence $\{n_i\}_{i=1}^{\infty} \prod_{i=1}^{\infty} l_{2m}^{\infty}$ *as follows*:

$$n_i = (Z_i, Z_i, Z_i, \ldots) \qquad \text{for } i = 1, 2, \ldots, n,$$

$$\eta_{n+1} = (\Phi_1, \Phi_2, \Phi_3, \ldots),$$

and

$$\eta_{n+k} = (\underbrace{0, 0, \ldots 0}_{(k-1)\text{times}}, \Psi_k, 0, 0, \ldots) \qquad \text{for } k = 2, 3, \ldots.$$

Then;

(1) $\eta_i \to^{\omega^*} 0$,

(2) $a_{n+1}(\{\eta_i\}_{i=1}^{\infty}) = 1$,

(3) $a_{n+1}(\{\eta_i\}_{i=1}^{\infty})$ *is not attained.*

Proof. (1) It is clear that $\eta_i \to^{\omega^*} 0$.

(2) For each positive integer $k \geqslant 1$, let $y_k = (\gamma, (1/k)\gamma) \in l_{2m}^{\infty}$, and define $y = (y_1, y_2, y_3, \ldots) \in \prod_{i=1}^{\infty} l_{2m}^{\infty}$. Let N_0 be the $(n+1)$-dimensional subspace of $\prod_{i=1}^{\infty} l_{2m}^{\infty}$ generated by $\{\eta_1, \ldots, \eta_n, y\}$. It will be shown that $d(\{\eta_i\}_{i=1}^{\infty}, c_0(N_0)) \leqslant 1$. Let $\varepsilon > 0$ be given, and let i_0 be a positive integer more than $1/\varepsilon$. Define the sequence $\{\tau_i\}_{i=1}^{\infty}$ in N_0 as follows:

$$\tau_i = \begin{cases} \eta_i & \text{for } i = 1, 2, \ldots, n, \\ y & \text{for } i = n+1, n+2, \ldots, i_0 \\ 0 & \text{for } i > i_0. \end{cases}$$

Obviously $\tau_i \to 0$, that is $\{\tau_i\}_{i=1}^{\infty} \in c_0(N_0)$, and

$$\begin{aligned} \|\{\eta_i\}_{i=1}^{\infty} - \{\tau_i\}_{i=1}^{\infty}\| &= \sup_i \|\eta_i - \tau_i\| = \sup_{i>n} \|\eta_i - \tau_i\| \\ &= \sup\{\{\|\eta_i - y\|; i = n+1, \ldots, i_0\} \cup \{\|\eta_i\|; i > i_0\}\} \\ &\leqslant 1 + \varepsilon. \end{aligned}$$

Since ε is arbitrary then $a_{n+1}(\{\eta_i\}_{i=1}^{\infty}) \leqslant d(\{\eta_i\}_{i=1}^{\infty}, c_0(N_0)) \leqslant 1$.

(3) Let N be an $(n+1)$-dimensional subspace of $\prod_{i=1}^{\infty} l_{2m}^{\infty}$, and assume that $d(\{\eta_i\}_{i=1}^{\infty}, c_0(N)) \leqslant 1$. It will be shown that if there is a sequence $\{\tau_i\}_{i=1}^{\infty}$ in N such that $\|\{\eta_i\}_{i=1}^{\infty} - \{\tau_i\}_{i=1}^{\infty}\| \leqslant 1$, the $\tau_i \not\to 0$. For

$x=(x_1, x_2,\ldots)\in\prod_{i=1}^{\infty} l_{2m}^{\infty}$ let $P_k\colon \prod_{i=1}^{\infty} l_{2m}^{\infty}\to l_{2m}^{\infty}$, $P_k(x)=x_k$, $k=1, 2,\ldots.$ Then for $k>1$

$$P_k(\eta_i)=Z_i \qquad \text{for } i=1, 2,\ldots, n,$$

$$P_k(\eta_{n+1})=\Phi_k,$$

and
$$P_k(\eta_{k+n})=\Psi_k.$$

So $\delta(\{Z_1, Z_2,\ldots, Z_n, \Phi_k, \Psi_k\}, P_k(N))\leqslant 1$. Since Φ_k and Ψ_k satisfy the conditions of Lemma 1.2, then $\|\tau_k\|\geqslant\|P_k(\tau_k)\|\geqslant 1$. To $\tau_i \nrightarrow 0$.

1.4. *Note.* For the case $n=1$, there is an easy example. Let $x_1=(2, 1, 1, 1,\ldots)\in l_\infty$,

$$x_i=\Bigl(\underbrace{0, 0,\ldots, 0}_{(i-1)\text{times}}, \frac{i+1}{i}, \frac{-i}{i+1}, 0, 0,\ldots\Bigr) \qquad \text{for } i=2, 3,\ldots.$$

Then $x_i\to^{\omega^*} 0$, and if N_0 is the 1-dimensional subspace of l_∞ generated by the element $y_0=(1, \frac{1}{2}, \frac{1}{3},\ldots)$, then

$$a_1(\{x_i\}_{i=1}^{\infty})\leqslant d(\{x_i\}_{i=1}^{\infty}, \qquad c_0(N_0))\leqslant 1.$$

Furthermore if $y\in l_\infty$ and N is the 1-dimensional subspace generated by y, one can show that if $\|x_i-\alpha_i y\|\leqslant 1$, $i=1, 2,\ldots$, then $\alpha_i\nrightarrow 0$.

1.5. Theorem. *For any positive integer $n\geqslant 1$, the set $K_n(l_1, c_0)$ is not proximinal in $L(l_1, c_0)$.*

Proof. By Lemma 0.1, (taking Q to be the set of positive integer), $K_n(l_1, c_0)$ is proximinal in $L(l_1, c_0)$, iff for each bounded sequence $\{x_i\}_{i=1}^{\infty}$ in l_∞ with $x_i\to^{\omega^*} 0$, there is an n-dimensional subspace N of l_∞, and a sequence $\{\tau_i\}_{i=1}^{\infty}$ in $c_0(N)$, such that $a_n(\{x_i\}_{i=1}^{\infty})=\|\{x_i\}_{i=1}^{\infty}-\{\tau_i\}_{i=1}^{\infty}\|$. By Lemma 1.3 and note 1.4 this is not true. ▌

Theorem 1.5 gives a negative solution for the Problem 5.2.1 in Deutsch *et al.* [2], and since by Mach and Ward [5], the set $K(l_1, c_0)$ is proximinal in $L(l_1, c_0)$, it gives a positive solution for the Problem 5.2.4 in the same paper.

2. $K_n(l_1, C(Q))$ Is Not Proximinal in $K(l_1, C(Q))$ If Q Contains Q_0

In this section it will be shown that if Q is a compact Hausdorff space Q that contains Q_0, then for each positive integer $n\geqslant 1$, the set $K_n(l_1, C(Q))$ is not proximinal in $K(l_1, C(Q))$. It will be shown also that for each positive integer $n \geqslant 1$, there is a bounded subset A of c_0, such that the n-width of A, $d_n(A, c_0)$ is not attained.

The proof of the following lemma can be found in Feder [3]:

2.1. LEMMA. *Let $T: l_1 \to E$ be a bounded linear operator from l_1 into any Banach space E, and let B_{l_1} be the closed unit ball of l_1. Then*

(1) $a_n(T) = d_n(T(B_{l_1}), E)$

(2) *$a_n(T)$ is attained iff $d_n(T(B_{l_1}, E)$ is attained.*

2.2. COROLLARY. *Let E be a Banach space and let n be any non-negative integer. Then the set $K_n(l_1, E)$ is proximinal in $L(l_1, E)$ [resp. $K(l_1, E)$] iff $d_n(A, E)$ is attained for any countable bounded [resp. relatively compact] subset A of E*

Proof. If $A = \{x_1, x_2, ...\} \subseteq E$ then let $T: l_1 \to E$ be the linear operator defined by $T(e_i) = x_i$, $i = 1, 2, ...$, where $\{e_i\}_{i=1}^{\infty}$ is the standard basis in l_1. Clearly T is bounded [resp. compact] if A is bounded [resp. relatively compact], thus the result follows from Lemma 2.1.

2.3. DEFINITION. Let Q be a locally compact Hausdorff space, Q will be said to "contain Q_0" if it contains a subset that is homeomorphic to the one point compactification of the set of positive integers; that is to say if it contains an infinite convergent sequence of distinct elements. It is obvious that if Q does not contain Q_0, then every subset Y of Q does not contain Q_0. If, in the Hausdorff space Q, there is a nonisolated element b_0 that has a countable basis of neighborhoods, then Q contains Q_0. Thus every first coubtable nondiscrete Hausdorff space contains Q_0. Furthermore, there are separable compact infinite Hausdorff spaces, that do not contain Q_0. Indeed if βN is the Stone–Cech compactification of the set of positive integers, then every subspace of βN does not contain Q_0.

2.4. PROPOSITION. *Brown* [1]; *If Q is a compact Hausdorff space such that there is the $C(Q)$ satisfying;*

$$\overline{\{q \in Q; h(q) < 0\}} \cap \overline{\{q \in Q; h(q) > 0\}} \neq \phi,$$

then for any positive integer $n \geqslant 1$, there is a bounded subset A of $C(Q)$ lying in a $(n+3)$-dimensional subpace of $C(Q)$, such that $d_n(A, C(Q))$ is not attained.

2.5. LEMMA. *Let Q be a compact Hausdorff space that contains Q_0. There is a continuous function $h \in C(Q)$ satisfying;*

$$\overline{\{q \in Q; h(q) < 0\}} \cap \overline{\{q \in Q; h(q) > 0\}} \neq \phi$$

Proof. Let $b_k \to b_0$ be an infinite convergent sequence in Q, such that $b_i \neq b_j$ when $i \neq j$. Define $f\colon \{b_k\}_{k=1}^{\infty} \cup \{b_0\} \to R$ by

$$f(b_k) = \begin{cases} \frac{1}{k} & \text{if } k \text{ is even,} \\ -\frac{1}{k} & \text{if } k \text{ is odd,} \\ 0 & \text{if } k = 0. \end{cases}$$

Then f is continuous. Since Q is compact Hausdorff space and $\{b_k\}_{k=1}^{\infty} \cup \{b_0\}$ is a closed subset of Q, then by Tietze extension Theorem, there is a continuous function $h \in C(Q)$ such that $h|_{\{b_k\}_{k=1}^{\infty} \cup \{b_0\}} = f$. It is obvious that $b_0 \in \overline{\{q \in Q; h(q) < 0\}} \cap \overline{\{q \in Q; h(q) > 0\}}$.

2.6. COROLLARY. *If Q is a compact Hausdorff space that contains Q_0 then for each positive integer $n \geqslant 1$, the set $K_n(l_1, C(Q)$ is not proximinal in $K(l_1, C(Q))$.*

Proof. By Proposition 2.4 and Lemma 2.5, for each $n \geqslant 1$, there is a bounded set A of $C(Q)$, such that A lies in an $(n+3)$-dimensional subspace of $C(Q)$ and $d_n(A, C(Q))$ is not attained. Since A lies in a finite dimensional subspace of $C(Q)$, it follows that A is relatively compact and separable, so the result follows from Corollary 2.2.

2.7. COROLLARY. *For any positive integer $n \geqslant 1$, there is a bounded subset A of c_0 such that $d_n(A, c_0)$ is not attained.*

Proof. Follows from Theorem 1.5 and Corollary 2.2.

2.8. COROLLARY. *For any relatively compact subset A of c_0 and any non-negative integer $n \geqslant 0$, the n-width $d_n(A, c_0)$ is attained.*

Proof. By Deutsch *et al.* [2], the set $K_n(l_1, c_0)$ is proximinal in $K(l_1, c_0)$, thus by Corollary 2.2 for any relatively compact subset A of c_0 and any non-negative integer $n \geqslant 0$ the n-width $d_n(A, c_0)$ is attained. ∎

Corollary 2.6 is the first example in which the set $K_n(X, Y)$ is not proximinal in $K(X, Y)$. Corollary 2.7 cannot be obtained from Proposition 2.4, because c_0 is not isometric to any $C(Q)$, for which Q satisfies the condition of the proposition, indeed Corollary 2.8 shows that the proposition is not true if $C(Q)=c_0$.

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